NON-LINEAR HYBRID CONTROL ORIENTED MODELLING OF A DIGITAL DISPLACEMENT MACHINE

Niels H. Pedersen, Per Johansen, Torben O. Andersen, Rudolf Scheidl*
Department of Energy Technology
Aalborg University, Johannes Kepler University*
Pontoppidanstraede 111, 9220 Aalborg East, Denmark
E-mail: nhp@et.aau.dk, pjo@et.aau.dk, toa@et.aau.dk, rudolf.scheidl@jku.at*

ABSTRACT

Proper feedback control of digital fluid power machines (Pressure, flow, torque or speed control) requires a control oriented model, from where the system dynamics can be analyzed, stability can be proven and design criteria can be specified. The development of control oriented models for hydraulic Digital Displacement Machines (DDM) is complicated due to non-smooth machine behavior, where the dynamics comprises both analog, digital and non-linear elements. For a full stroke operated DDM the power throughput is altered in discrete levels based on the ratio of activated pressure chambers. In this paper, a control oriented hybrid model is established, which combines the continuous non-linear pressure chamber dynamics and the discrete shaft position dependent activation of the pressure chambers. The hybrid machine model is further extended to describe the dynamics of a Digital Fluid Power Transmission (DFPT) comprising two variable speed DDM's with asynchronous control sampling schemes. A validation with respect to a non-linear dynamical model representing the physical system, shows the usefulness of the hybrid model with respect to feedback control development.

KEYWORDS: Fluid Power, Digital Displacement, Hybrid system, Control model, Event-driven

1 INTRODUCTION

Digital fluid power is gaining an increased interest as an alternative to its conventional analogue counterpart due to its fast dynamic response and high efficiency [1, 2]. Digital fluid power machines are characterized by delivering a discrete volumetric output determined by the fraction of activated pressure chambers. Fluid power transmissions (drives) is one application with a high potential for the use of these digital hydraulic machines, since robustness, repeatability and redundancy is an appreciated property that digital fluid power brings [3].

The so called digital displacement machine (DDM) is of this type and is the research topic for a high number of papers focusing on design and performance optimization [4, 5, 6, 7, 8, 9, 10, 11]. The machine comprises of several cylinders being radially distributed around a common shaft (eccentric or cam-ring type). The pressure in each of these cylinder chambers is controlled by electrically actuated on/off valves connected to a common high and low-pressure manifold. Due to these digital valves, the displacement throughput of a full stroke operated DDM cannot be altered continuously, but rather in a stroke-by-stroke fashion once for every reciprocating piston motion. In a full stroke operation, each pressure chamber is either active or inactive during a full stroke, which means that a pump chamber is either pumping or idling and a motor chamber is either motoring or idling. This inherent feature complicates the control of the machine, which is of high importance for obtaining high efficiency and performance with respect to torque ripples and reference tracking.

Control development for digital displacement machines is a complicated task and requires a control oriented model, such that proper feedback controllers can be designed. The development of such dynamical model is not straightforward since the system comprises several dynamically aspects. The dynamics of the machine and its pressure chambers is governed by non-linear continuous differential equations, while the input is binary (active or inactive) and may only be updated discretely at certain fixed shaft angles. Additionally, when an actuation decision (active or inactive) has been made for a cylinder chamber, the output is affected for the next half of the stroke, during where the mentioned actuation decision cannot be changed.

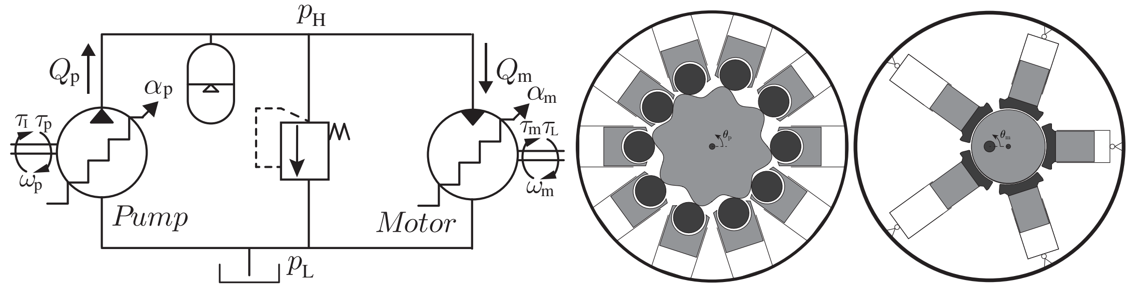
No traditionally control oriented models are able to capture all the mentioned effects, but is instead limited to describe some of these effects or use approximations of them. Both globally asymptotically stabilizing control feedbacks for continuous and discrete linear systems has been widely studied and various approaches exist [12]. In the case of a non-linear sample-and-hold control system, the designer often tend to approximate the system dynamics, often though linearization and discretization, which puts a limiting factor on the obtainable performance [13, 14]. Another obstacle in the control of a DDM, is the position triggered sampling, which contrarily to time triggered sampling is not widely used and therefore remains an undergoing research topic. However, methods of state dependent event-driven sampled control exist, but is in general limited to linear system [15, 16, 17] or focused around minimizing the number of events in networked control systems [18, 19].

Control development for digital displacement machines is an expanding research topic and several results on the subject has been published. Most of these results are however limited to simplified operation conditions under fixed speed and often in an open loop configuration [5, 20, 21, 22]. Promising results regarding feedback control have been published by Sniegucki et. al. [23], which utilizes mixed logical dynamic programming to determine the optimal actuation sequence. Despite yielding promising results with respect to torque fluctuations and tracking, its use has so far been limited to offline optimization due to a large computational effort. Johansen et. al [24] proposes a closed loop structure for fixed speed operation, where the digital control signal is converted to a binary actuation sequence by a Delta-Sigma modulator. The same authors [25] presents a discrete linear time invariant (DLTI) model approximation of the digital machine, allowing for closed loop control development for a fixed speed machine. The work of these two papers has been combined by Pedersen et. al [26, 27] and applied to a digital fluid power transmission for a wind turbine. An event-driven control strategy using the DLTI model and Delta-Sigma modulator to overcome the problem of variable speed operation is presented in [28]. However, all of these approaches are based on discrete linear approximations, which puts limitations to the performance and region of stability. Furthermore, no control oriented model for a DFPT with two DDM's has so far been developed.

In this study, a hybrid representation of a digital displacement machine is made, followed by a hybrid model of a digital fluid power transmission. Hybrid dynamical models are able to capture both discrete and continuous non-linear effects, as well as irrational state dependent sampling [29, 30]. Furthermore, the theory of hybrid dynamical systems has been well studied in the recent years, which has provided tools for designing globally asymptotically stabilizing feedbacks [29, 31]. This paper is structured as follows, initially as a foundation to deriving the hybrid model, the description of the DFPT is given together with the operation principle of the digital pump and motor. Following this a brief introduction to hybrid system theory in the case of sample and hold systems is provided. Then the hybrid representation of the DDM is given and verified by comparison to a non-linear dynamic simulation model representing the physical system. Lastly, the full DFPT hybrid model is derived and verified.

2 SYSTEM DESCRIPTION

The Digital Fluid Power Transmission (DFPT) under consideration comprises a multi-lobe cam-ring type digital displacement pump and a radial piston type digital displacement motor connected through a high and low pressure line as illustrated in Fig. 1.



(a) Illustration of the Digital Fluid Power transmission system.

(b) Design illustration of the digital displacement pump (left) and motor (right).

Figure 1: Sketch of a digital fluid power transmission and the pump/motor designs.

The DFPT is shown in a general set-up and is applicable as transmission in e.g. wind turbines, where the turbine rotor drives the pump and the motor is connected to an electrical generator. An input torque τ_I is seen to drive the DFP pump which outputs a pressurized fluid flow Q_p to the high-pressure line having the pressure p_H. The DFP motor intakes a flow Q_m from the high-pressure line and converts it to a shaft rotation through the mechanical torque τ_m. In general, both the pump and motor may be operating at variable speed and torque/flow, which is controlled by the displacement input α_p and α_m for the pump and motor respectively. In this specific case, the machines are dimensioned around a 5 MW transmission. The pump having the design principle illustrated to the left in Fig. 1b is a cam-ring type with radially distributed cylinder pistons attached by use of roller bearings. In correct scale, there are 16 ring cam lobes and 25 cylinders in one module and 4 modules in total, which are radially shifted to avoid simultaneous chamber activations. As a result, the pump consists of 100 cylinders, each going through 16 strokes for every pump revolution. The lobed geometry yields a high displacement and a great flow/torque output resolution for a low speed machine (< 15 rpm) with a compact design. The motor having the design principle illustrated to the right, is a radial piston type with an eccentric cam connected to a common shaft. In correct scale 7 cylinders are distributed radially around the eccentric cam shaft. A total of 6 modules are radially shifted around the common shaft similar to the pump, such that there is a total of 42 cylinders doing one stroke each for every motor revolution. This design is suited for a high-speed machine (∼1500 rpm) with a relative low displacement per revolution.

Digital displacement machines are characterized by having fast switching electrically actuated on/off valves controlling the flow throughput of the numerous cylinder pressure chambers. A sketch of a single motor cylinder pressure chamber is illustrated in Fig. 2, and is essentially identically for the pump design.

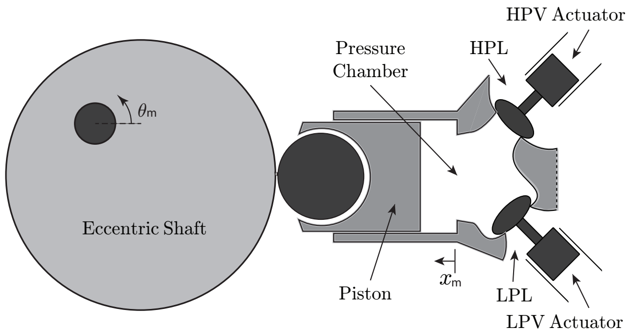
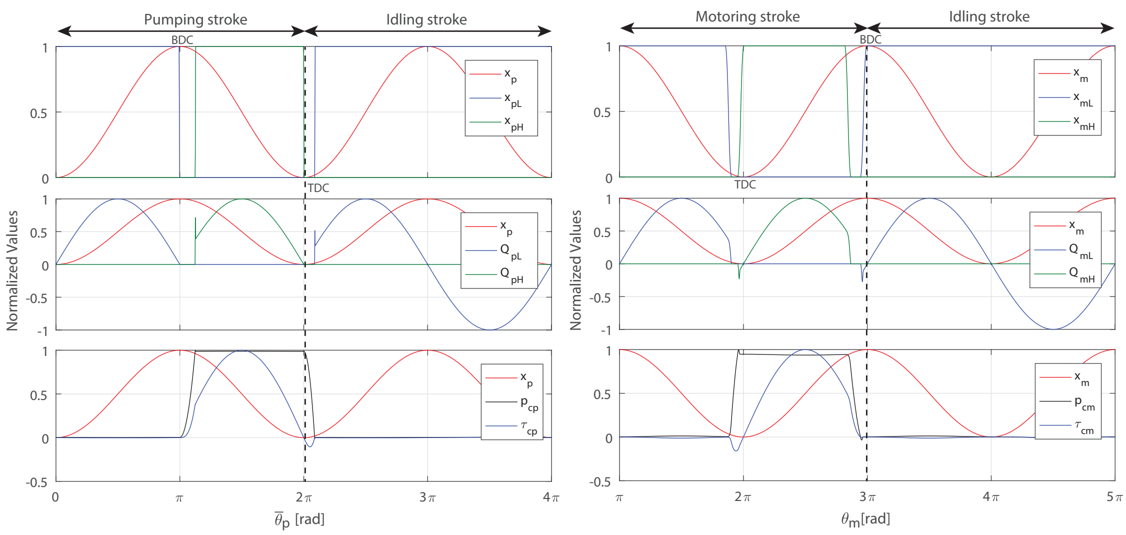


Figure 2: Sketch of a single motor pressure chamber.

Proper operation of the machine requires precisely timed actuation of the high-pressure valve (HPV) and low-pressure valve (LPV). Essential to the underlying framework of developing a control oriented model of the digital displacement machines is the understanding of the operating principle. Considering a full-stroke operation there are three distinct operating modes being idling, pumping and motoring. In a full stroke operation, the mode of each displacement chamber is altered on a stroke-by-stroke basis. Simulation results showing a pumping, idling and motoring stroke is shown in Fig. 3 and is obtained from a non-linear simulation model of the digital displacement machines presented in [26, 28].



(a) Pumping and idling mode.

(b) Motoring and idling mode.

Figure 3: Simulation results of a pumping, idling and motoring stroke of the digital displacement machines.

The local pumping angle is defined as $\bar{\theta}_{\mathrm{p}} = \theta_{\mathrm{p}} N_{\mathrm{p,l}}$, where $N_{\mathrm{p,l}} = 16$ is the number of cam lobes. For the pumping mode, the LPV is closed just ahead of Buttom Dead Center (BDC), such that a passive opening of the HPV is obtained due to the chamber pressure exceeding the high pressure. During the active part of the pumping stroke, pressurized fluid is pumped to the high-pressure line. Near Top Dead Center (TDC), the HPV is closed and the LPV is passively opened since the chamber pressure drops below the low pressure. The motoring stroke is seen to be essentially the opposite of the pumping stroke, where pressurized fluid is drawn from the high-pressure line.

3 CONTROL ORIENTED MODEL

Based on the system description, the following things about the digital displacement machines may be stated with respect to development of a control oriented model.

- The dynamics of each pressure chamber is described by non-linear continuous differential equations.
- The actuation input to each pressure chamber is binary (active or inactive).
- An update of the actuation input can only be made at a fixed shaft angle.
- After an actuation input for a pressure chamber has been made, the output is affected for almost half of a piston revolution.

With respect to developing a control oriented model, the system contains non-linear dynamics, as well as both continuous and discrete elements. Additionally, memory effects are necessary, since the output depends on the input decisions after numerous subsequent input decisions. As a result, classical continuous and discrete control methods are not adequate, why a hybrid model is established to include all these different effects. Hybrid theory is very broad and is able to describe most known dynamical processes by combining the analog and digital world.

3.1 SAMPLE AND HOLD HYBRID SYSTEM THEORY

A hybrid system comprises both continuous differential (flow) equations and discrete difference (jump) equations. A hybrid system is in general formulated as that given by (1) in accordance with [32, 30].

$$\mathscr{H}: \quad \begin{cases} \dot{x} & \in F(x,u), \qquad x \in C, \\ x^+ & \in G(x,u), \qquad x \in D \\ y & = h(x,u) \end{cases} \tag{1}$$

$\dot{x}$ denotes the state time derivative and x^+ denotes the state value after a jump. u and y are system input and output respectively. For simplicity, a hybrid system is written as $\mathscr{H} = (F,C,G,D,h)$, where the variables describes the following

- The flow set: C
- The flow map: F
- The jump set: D
- The jump map: G
- The output map h

As long as x belongs to the flow set C, x is described by the differential inclusion given by the flow map F and when x belongs to the jump set D, x is described by the difference inclusion given by the jump map G.

To illustrate the use of hybrid dynamical system theory, a sample-and-hold control system where a continuous plant is feedback controlled by a digital controller may be formulated as a relatively simple hybrid control system given by (2) [32, 30].

Flow map and set:

$$\left.\begin{aligned} \dot{x} &= f(x,u) \\ \dot{\tau} &= 1 \\ \dot{z} &= 0 \end{aligned}\right\} \qquad \tau \in [0, T_{\mathrm{s}}] \tag{2}$$

Jump map and set:

$$\left.\begin{aligned} x^{+} &= x \\ \tau^{+} &= 0 \\ z^{+} &= u = \mathscr{C}_{\kappa}(z,x) \end{aligned}\right\} \qquad \tau = T_{\mathrm{s}} \tag{3}$$

Where τ is a sawtooth shaped sampling timer that tracks the time since the last sample and resets when a sample occurs at the sampling time T_{s}. z is the internal controller state, which here represent the zero-order hold input to the plant, u. $\mathscr{C}_{\kappa}$ is the feedback stabilizing control law. An illustration of a simple sample-and-hold system is given in (4) and the response is shown in Fig. 4. where $\begin{bmatrix} x(0) & \tau(0) & z(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

$$\begin{aligned} &\left.\begin{aligned} \dot{x} &= -0.1\,x + u \\ \dot{\tau} &= 1 \\ \dot{z} &= 0 \end{aligned}\right\} \qquad \tau \in [0, 1] \\ &\left.\begin{aligned} x^{+} &= x \\ \tau^{+} &= 0 \\ z^{+} &= u = -0.4\,x \end{aligned}\right\} \qquad \tau = 1 \end{aligned} \tag{4}$$

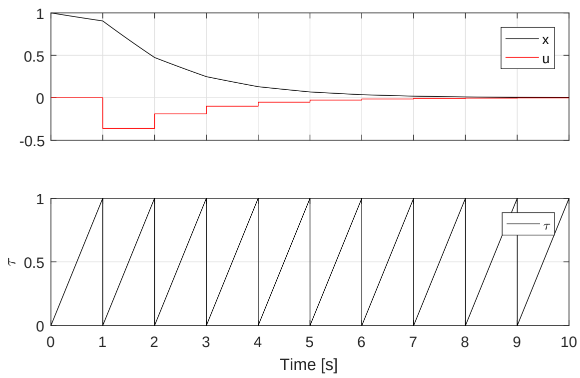


Figure 4: Response of hybrid dynamical sample-and-hold system.

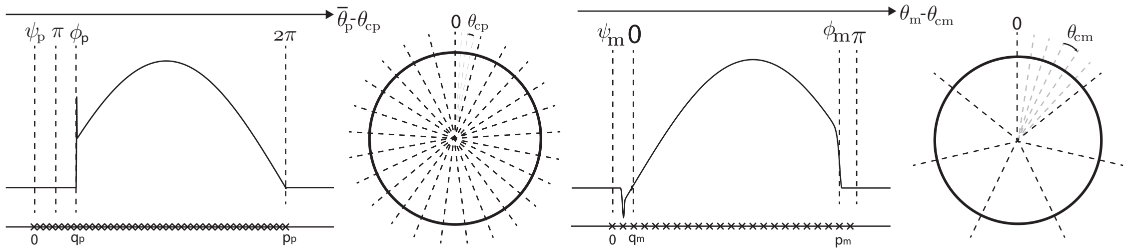
It is seen that the timer state τ is reset when the timer reach the jump map $T_{\mathrm{s}} = 1$, where the input $u = z$ is updated. Between jumps the continuous state evolution occurs in correspondence with the flow map. This method of modeling a sample-and-hold feedback control system is exploited, when modeling the digital displacement machine control system.

3.2 HYBRID FORMULATION OF A DDM

The displacement volume and its derivative of a single pressure chamber may be described by (5) [26, 27, 28].

$$\begin{aligned} V\left(\theta(t)\right) &= k_{\mathrm{q}}\left(1 - \cos(\theta(t))\right) \\ \dot{V}\left(\theta(t)\right) &= k_{\mathrm{q}}\,\omega(t)\,\sin(\theta(t)) \end{aligned} \tag{5}$$

where $k_\mathrm{q} = \hat{V}/2$, $\omega = \dot{\theta}$ and $\hat{V}$ being the maximum displacement volume. In this study, it is valid to neglect the pressure dynamics during a single stroke as seen in Fig. 3, why the following approximation is valid $Q \approx \dot{V}$. The hybrid model for approximating the flow throughput of the DDM is derived based on the illustrations shown in Fig. 5.



(a) Pump. $\theta_\mathrm{cp} = 2\pi/N_\mathrm{p} = 2\pi/100$ (b) Motor. $\theta_\mathrm{cm} = 2\pi/N_\mathrm{m} = 2\pi/42$

Figure 5: Definitions of angles used to derive the hybrid model.

θ_cp and θ_cm are the angle between each cylinder on the common shaft. ψ_p is the pump decision angle, where the decision to use a pumping or an idling stroke is made and ψ_m is the motor decision angle, where the decision to use a motoring or an idling stroke is made. ϕ_p indicates where the active part of the pump stroke is initialized and ϕ_m indicates where the active part of the motoring stroke ends. Since the derivation is very similar for the pump and motor, the hybrid approximation to the flow throughput is only derived for the motor in this paper. In a full stroke operation, the motor flow throughput may be described as

$$Q_\mathrm{mH}(t) \approx \begin{cases} k_\mathrm{qm}\, \omega_\mathrm{m}(t)\, \sin(\theta_\mathrm{m}(t) - \theta_\mathrm{cm}) & if \;\; 0 < \theta_\mathrm{m}(t) - \theta_\mathrm{cm} \leq \phi_\mathrm{m} \\ 0 & else \end{cases} \tag{6}$$

If a decision of a motoring stroke is made, the active stroke part ends almost half a revolution later, where almost $N_\mathrm{m}/2$ numbers of decisions for other cylinders has been made meanwhile. Therefore, a memory state for each input decision in the active part of the motoring stroke is necessary. Additionally, a memory state of the shaft angle θ_m where the sampling occurred is required to describe the flow throughput as function of the shaft angle. As a result an input decision and an angle sampling state vector is defined by

$$z_{mu} = \begin{bmatrix} z_{mu,0} & z_{mu,1} & \cdots & z_{mu,pm} \end{bmatrix}^T \quad z_{m\theta} = \begin{bmatrix} z_{m\theta,0} & z_{m\theta,1} & \cdots & z_{m\theta,pm} \end{bmatrix}^T \tag{7}$$

Since the motoring stroke starts at an angle of $\theta_\mathrm{m} = 0$ and the decision angle ψ_m is located ahead of this, the shaft angle memory state is shifted by this delay angle and is defined as $z^+_{m\theta,i} = \theta_\mathrm{m} - \psi_\mathrm{m}$ The flow throughput of the machine can then be written by use of Fig. 5a and combining (6) and (7). The resulting motor flow model is given by

$$Q_\mathrm{m}(t) = \sum_{i=1}^{N_\mathrm{m}} Q_\mathrm{mH,i}(t) \approx k_\mathrm{qm}\, \omega_\mathrm{m}(t) \sum_{i=q_\mathrm{m}}^{p_\mathrm{m}} \sin\left(\theta_\mathrm{m}(t) - z_{m\theta,i}\right) z_{mu,i} \tag{8}$$

Similarly, it may be derived that the torque throughput can be described by

$$\tau_\mathrm{m}(t) = \sum_{i=1}^{N_\mathrm{m}} \tau_\mathrm{cm,i}(t) \approx k_{\tau m}\, p_\mathrm{H}(t) \sum_{i=q_\mathrm{m}}^{p_\mathrm{m}} \sin\left(\theta_\mathrm{m}(t) - z_{m\theta,i}\right) z_{mu,i} \tag{9}$$

Where k_qm and $k_{\tau m}$ are the flow and torque coefficients. It is evident, that the precision of the model will suffer for a low number of cylinders, which will be elaborated on based on a verification of the presented hybrid model.

3.3 HYBRID DDM MODEL VALIDATION

A validation of the hybrid model is made by comparing the impulse response for the flow and torque of the hybrid model to that of the non-linear dynamical model shown in Fig. 3. To simplify the verification process the following sample and hold hybrid model has been used

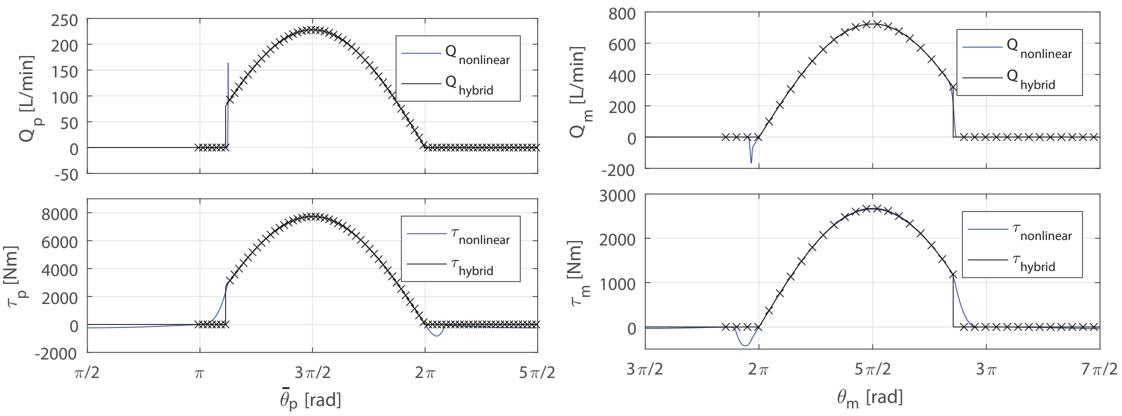
$$\left.\begin{aligned} \dot{\theta}_{\mathrm{m}} &= \omega_{\mathrm{m}} \\ \dot{\chi}_{\mathrm{m}} &= \omega_{\mathrm{m}} \\ \dot{z}_{\mathrm{mu,0}} &= 0 \\ \dot{z}_{\mathrm{mu,i}} &= 0 \;\; i = [1,..,p_{\mathrm{m}}] \\ \dot{z}_{m\theta} &= 0 \\ \dot{z}_{m\theta,\mathrm{i}} &= 0 \;\; i = [1,..,p_{\mathrm{m}}] \end{aligned}\right\} \chi_{\mathrm{m}} \in [0, \theta_{\mathrm{cm}}] \quad \begin{aligned} \theta_{\mathrm{m}}^{+} &= \theta_{\mathrm{m}} \\ \chi_{\mathrm{m}}^{+} &= 0 \\ z_{\mathrm{mu,0}}^{+} &= u_{\mathrm{m}} \\ z_{\mathrm{mu,i}}^{+} &= z_{\mathrm{mu,i\text{-}1}} \;\; i = [1,..,p_{\mathrm{m}}] \\ z_{m\theta,0}^{+} &= \theta_{\mathrm{m}} - \psi_{\mathrm{m}} \\ z_{m\theta,\mathrm{i}}^{+} &= z_{m\theta,\mathrm{i\text{-}1}} \;\; i = [1,..,p_{\mathrm{m}}] \end{aligned}\left.\vphantom{\begin{aligned}a\\a\\a\\a\\a\\a\end{aligned}}\right\} \chi_{\mathrm{m}} = \theta_{\mathrm{cm}} \tag{10}$$

where $y = \begin{bmatrix} Q_{\mathrm{m}} & \tau_{\mathrm{m}} \end{bmatrix}^T$ are calculated by use of (8) and (9). It is evident that the sampling is triggered by the position instead of time as done in conventional sample-and-hold systems. In this validation example with constant rotation speed, the sampling time is constant and the conventional representation could might as well had been used. However, for a variable speed DDM the sampling time is no longer constant but sampling angle is. Therefore, this representation is made to also allow for modeling of speed variable machines.

A description of the sample-and-hold scheme is made below for clarification

- The triggering state reaches $\chi_{\mathrm{m}} = \theta_{\mathrm{cm}}$ and the jump map is triggered. In the jump map the triggering state is reset: $\chi_{\mathrm{m}}^{+} = 0$.
- A motor actuation decision is made, determine whether the current cylinder pressure chamber should be idling $z_{\mathrm{mu,0}}^{+} = 0$ or motoring $z_{\mathrm{mu,0}}^{+} = 1$.
- The previous actuation decisions are shifted a single place, such that the newest decisions states are available.
- The current decision angle θ_{m} is sampled, $z_{m\theta,0}^{+} = \theta_{\mathrm{m}} - \psi_{\mathrm{m}}$. Lastly, the previous decision angles are shifted a single place.

The resulting impulse responses are shown in Fig. 6a and Fig. 6b for the pump and motor respectively. The system has been simulated by use of the Matlab toolbox "Hybrid Equations Toolbox v2.04" by R. Sanfelice. The values of the various parameters used in the model is provided in Tab. 1.



(a) Pump (b) Motor

Figure 6: Validation of hybrid flow and torque model of the DDM.

It is seen that there is a clear coherence between the non-linear model and the hybrid model. However, there are discrepancies at the end of the active stroke part due to valve and pressure dynamics. Since these differences are of relatively minor importance and are complex to model accurately, the hybrid formulation is accepted for system analysis and control development. As mentioned, the model accuracy will suffer for a low number of pressure chambers, since the discretization level will be poor. In that case, the model could be improved by including a different function than the proposed sinus function for the first and last sample of the active stroke part to increase the model precision.

3.4 Hybrid model for a DFPT

Developing a control oriented model for a digital fluid power transmission comprising both a digital displacement motor and pump is considered very problematic with conventional techniques. Considering the DLTI event-driven control strategy proposed by Pedersen et. al. [28] for speed varying digital displacement machines, it is not possible to extend the model to include two machines in a multi-rate sampling scheme, since the update rates are varying independently of each other. However, hybrid theory is able to overcome this problem by utilizing several jump maps.

A non-linear state equations for the continuous plant is given by (11) and is derived with basis in the definitions of variables shown in Fig. 1.

$$\underbrace{\begin{bmatrix} \dot{\theta}_\mathrm{m}(t) \\ \dot{\omega}_\mathrm{m}(t) \\ \dot{\bar{\theta}}_\mathrm{p}(t) \\ \dot{\omega}_\mathrm{p}(t) \\ \dot{p}_\mathrm{H}(t) \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} \omega_\mathrm{m}(t) \\ -\frac{d_\mathrm{m}}{J_\mathrm{m}}\,\omega_\mathrm{m}(t) + \frac{1}{J_\mathrm{m}}\,\tau_\mathrm{m}(t) - \frac{1}{J_\mathrm{m}}\,\tau_\mathrm{L}(t) \\ \omega_\mathrm{p}(t)\,N_\mathrm{p,l} \\ -\frac{d_\mathrm{p}}{J_\mathrm{p}}\,\omega_\mathrm{p}(t) - \frac{1}{J_\mathrm{p}}\,\tau_\mathrm{p}(t) + \frac{1}{J_\mathrm{p}}\,\tau_\mathrm{I}(t) \\ -\frac{\beta}{V_\mathrm{H}}\,k_\mathrm{L}\,p_\mathrm{H}(t) + \frac{\beta}{V_\mathrm{H}}\,Q_\mathrm{p}(t) - \frac{\beta}{V_\mathrm{H}}\,Q_\mathrm{m}(t) \end{bmatrix}}_{f(x,\zeta)} \tag{11}$$

Where $\zeta = \left[Q_\mathrm{m}(t)\;\tau_\mathrm{m}(t)\;Q_\mathrm{p}(t)\;\tau_\mathrm{p}(t)\right]^T$. τ_i and τ_L represents the input torque to the pump shaft and the load torque to the motor respectively. J_p and J_m are the inertia masses, while d_p and d_m are the damping coefficient of the pump and motor shaft, respectively. β is the effective bulk modulus, V_H is the high pressure line volume and k_L is the leakage coefficient. The hybrid sample-and-hold scheme for the flow map is given by

$$\left.\begin{aligned} \dot{x} &= f(x,\zeta) \\ \dot{\chi}_{\mathrm{m}} &= \omega_{\mathrm{m}} \\ \dot{z}_{\mathrm{mu}} &= 0 \\ \dot{z}_{m\theta} &= 0 \\ \dot{\chi}_{\mathrm{p}} &= \omega_{\mathrm{p}} N_{\mathrm{p,l}} \\ \dot{z}_{\mathrm{pu}} &= 0 \\ \dot{z}_{p\theta} &= 0 \end{aligned}\right\} \begin{aligned} \chi_{\mathrm{m}} &\in [0,\theta_{\mathrm{cm}}] \\ \chi_{\mathrm{p}} &\in [0,\theta_{\mathrm{cp}}] \end{aligned} \wedge \tag{12}$$

When both triggering states are below their threshold angle, the flow map is used and when one of them reaches the triggering value a control update is initiated by entering the respective jump map given by

$$\left.\begin{aligned} x^+ &= x \\ \chi_{\mathrm{m}}^+ &= 0 \\ z_{\mathrm{mu,0}}^+ &= u_{\mathrm{m}} \\ z_{\mathrm{mu,i}}^+ &= z_{\mathrm{mu,i\text{-}1}} \;\; i = [1,..,p_{\mathrm{m}}] \\ z_{m\theta,0}^+ &= \theta_{\mathrm{m}} - \psi_{\mathrm{m}} \\ z_{m\theta,\mathrm{i}}^+ &= z_{m\theta,\mathrm{i\text{-}1}} \;\; i = [1,..,p_{\mathrm{m}}] \\ \chi_{\mathrm{p}}^+ &= \chi_{\mathrm{p}} \\ z_{\mathrm{pu}}^+ &= z_{\mathrm{pu}} \\ z_{p\theta}^+ &= z_{p\theta} \end{aligned}\right\} \begin{aligned} \chi_{\mathrm{m}} &= \theta_{\mathrm{cm}} \\ \chi_{\mathrm{p}} &\in [0,\theta_{\mathrm{cp}}] \end{aligned} \wedge \quad \left.\begin{aligned} x^+ &= x \\ \chi_{\mathrm{m}}^+ &= \chi_{\mathrm{m}} \\ z_{\mathrm{mu}}^+ &= z_{\mathrm{mu}} \\ z_{m\theta}^+ &= z_{m\theta} \\ \chi_{\mathrm{p}}^+ &= 0 \\ z_{\mathrm{pu,0}}^+ &= u_{\mathrm{p}} \\ z_{\mathrm{pu,i}}^+ &= z_{\mathrm{pu,i\text{-}1}} \;\; i = [1,..,p_{\mathrm{p}}] \\ z_{p\theta,0}^+ &= \bar{\theta}_{\mathrm{p}} - (\pi - \psi_{\mathrm{p}}) \\ z_{p\theta,\mathrm{i}}^+ &= z_{p\theta,\mathrm{i\text{-}1}} \;\; i = [1,..,p_{\mathrm{p}}] \end{aligned}\right\} \begin{aligned} \chi_{\mathrm{m}} &\in [0,\theta_{\mathrm{cm}}] \\ \chi_{\mathrm{p}} &= \theta_{\mathrm{cp}} \end{aligned} \wedge \tag{13}$$

Based on (8) the flow throughput of the machines may be approximated by

$$\begin{aligned} Q_{\mathrm{m}}(t) &= k_{\mathrm{qm}}\,\omega_{\mathrm{m}}(t) \sum_{i=q_{\mathrm{m}}}^{p_{\mathrm{m}}} \sin\left(\theta_{\mathrm{m}}(t) - z_{m\theta,i}\right) z_{mu,i} \\ Q_{\mathrm{p}}(t) &= k_{\mathrm{qp}}\,N_{\mathrm{p,l}}\,\omega_{\mathrm{p}}(t) \sum_{i=q_{\mathrm{p}}}^{p_{\mathrm{p}}} \sin\left(\bar{\theta}_{\mathrm{p}}(t) - z_{p\theta,i}\right) z_{pu,i} \end{aligned} \tag{14}$$

Similarly based on (9), it may be derived that the torque throughput can be described by

$$\begin{aligned} \tau_{\mathrm{m}}(t) &= k_{m\tau}\,p(t) \sum_{i=q_{\mathrm{m}}}^{p_{\mathrm{m}}} \sin\left(\theta_{\mathrm{m}}(t) - z_{m\theta,i}\right) z_{mu,i} \\ \tau_{\mathrm{p}}(t) &= k_{p\tau}\,N_{\mathrm{p,l}}\,p(t) \sum_{i=q_{\mathrm{p}}}^{p_{\mathrm{p}}} \sin\left(\bar{\theta}_{\mathrm{p}}(t) - z_{p\theta,i}\right) z_{pu,i} \end{aligned} \tag{15}$$

Hereby a dynamic hybrid system model of a digital fluid power transmission comprising two digital displacement machines is developed and may be used for dynamic analysis and feedback control development.

3.5 HYBRID DFPT MODEL VALIDATION

To verify the proposed hybrid formulation of the DFPT, a simulation of the model is made and the results are compared to those obtained by simulation of the non-linear model representing the physical transmission. The equations for the non-linear model are available

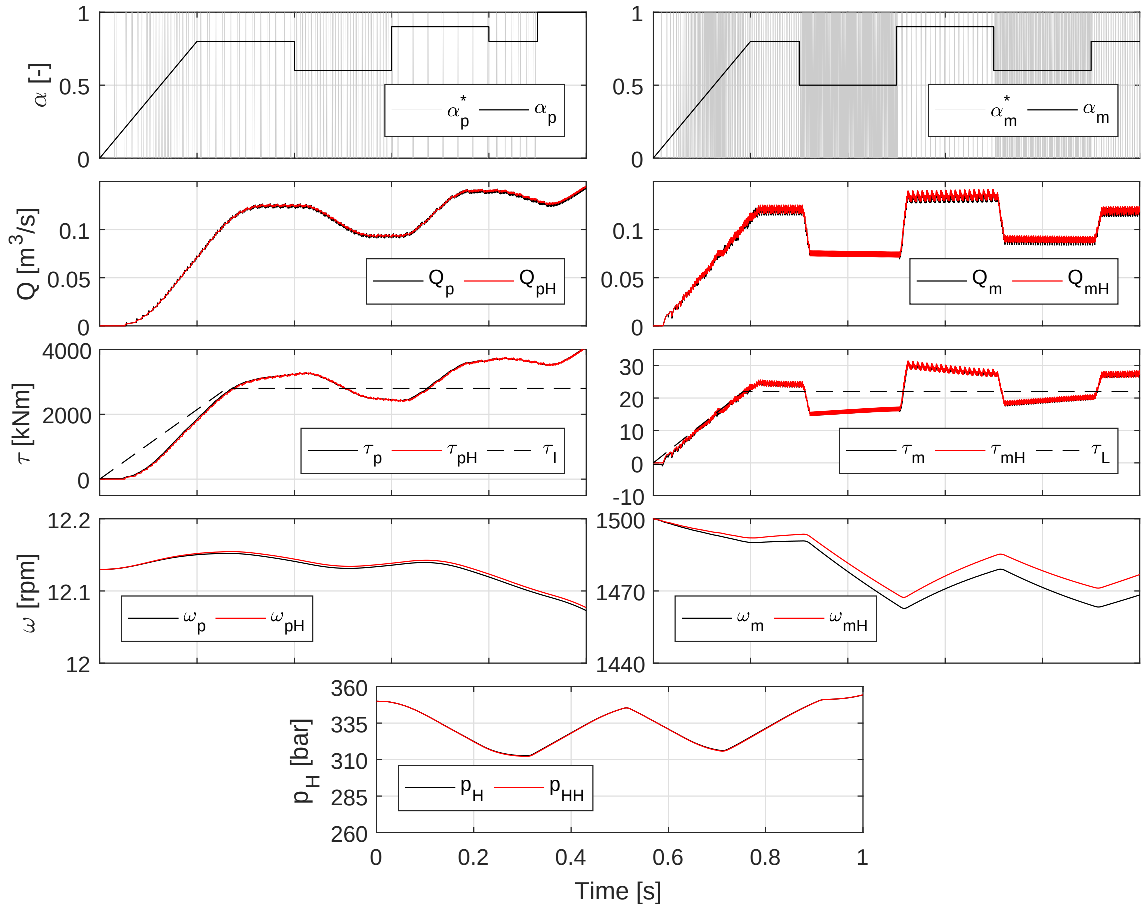


Figure 7: Results for validation of the hybrid DFPT model. Black lines represents the non-linear physical model and red lines represents the hybrid model.

in [26, 28], and the results for a single stroke of the pump and motor has already been shown in Fig. 3. In the validation, both models are given the same inputs (α_p, α_m, τ_I, τ_L). The displacement fractions α_p and α_m are converted to Pulse Width Modulated (PWM) signals α_p^* and α_m^* by a first order discrete Delta-Sigma-Modulator (DSM) [33] describing the cylinder chamber actuation sequence. The results are shown in Fig. 7, where it should be noted that only the generated PWM signals for the non-linear model are shown for clarity. The results are seen to have a great correspondence, which may be expected based on the validation of a single stroke shown in Fig. 6. The only easily identified difference is seen in the motor velocity response where drifting between the two responses is observed, which is due to the low motor inertia mass and damping combined with the minor discrepancy in the motor torque response. Overall, all the relevant system dynamics is captured with the hybrid model, which verifies its applicability with respect to system analysis and feedback control development.

4 CONCLUSION

To enhance the development of feedback stabilizing controllers for hydraulic digital displacement machines, a hybrid dynamical control oriented model of the system dynamics has been derived and verified. The presented digital displacement machine hybrid model avoids the previous model approximations by linearization, allowing for the possibility of an increased performance and enlarged region of stability. Hybrid system theory has also been proven to be able to describe the dynamics of a digital fluid power transmission system with multiple speed varying digital displacement machines, which is not possible with classical discrete or continuous methods. The next step is thus to design globally asymptotically stable feedback controllers based on well established tools within hybrid dynamical system theory.

5 APPENDIX

Table 1: Parameters values used in the model

Parameter	symbol	value	Unit	Parameter	symbol	value	Unit
Pump lobes in 1 module	$N_{\mathrm{p,l}}$	16	-	Pump flow constant	k_{qp}	249.3	cm^3
Pump cylinders in 1 module	$N_{\mathrm{p,c}}$	25	-	Pump torque constant	$k_{\tau p}$	249.3	cm^3
Number of pump modules	$N_{\mathrm{p,m}}$	4	-	Motor flow constant	k_{qm}	77	cm^3
Motor cylinders in 1 module	$N_{\mathrm{m,c}}$	7	-	Motor torque constant	$k_{\tau m}$	77	cm^3
Number of motor modules	$N_{\mathrm{m,m}}$	6	-	Pump damping coefficient	d_{p}	50e3	$N\,m\,s/rad$
Number of pump cylinders	N_{p}	100	-	Pump inertia mass	J_{p}	38.76e4	$kg\,m^2$
Number of motor cylinders	N_{m}	42	-	Motor damping coefficient	d_{m}	10	$N\,m\,s/rad$
Pump decision angle	ψ_{p}	178.8	deg	Motor inertia mass	J_{m}	534.2	$kg\,m^2$
Pump stroke angle	ϕ_{p}	18.2	deg	Effective bulk modulus	β	16000	bar
Motor decision angle	ψ_{p}	26.3	deg	High pressure line volume	V_{H}	3.92	m^3
Motor stroke angle	ϕ_{p}	161.8	deg				

6 ACKNOWLEDGMENT

This research was funded by the Danish Council for Strategic Research through the HyDrive project at Aalborg University, at the Department of Energy Technology (case no. 1305-00038B).

References

[1] W. Rampen, "Gearless transmissions for large wind turbines - the history and future of hydraulic drives," Artemis IP Ltd., Scotland.

[2] P. Silva, A. Giuffrida, and N. F. et. al., "Performance prediction of a multi-mw wind turbine adopting an advanced hydrostatic transmission," *Energy, Vol. 64, pp. 450 - 461*, December 2013.

[3] M. Linjama, "Digital fluid power state of the art," *The Twelfth Scandinavian International Conference on Fluid Power, May 18-20, Tampere, Finland*, 2011.

[4] G. S. Payne, U. P. P. Stein, M. Ehsan, N. J. Caldwell, and W. H. S. Rampen, "Potential of digital displacement hydraulics for wave energy conversion," *In Proc. of the 6th European Wave and TIdal Energy Conference, Glasgow UK.*, 2005.

[5] W. R. M. Ehsan and S. Salter, "Modeling of digital-displacement pump-motors and their application as hydraulic drives for nonuniform loads," *ASME, Journal of dynamic system measurement and control, Vol. 122, pp. 210-215*, March 2000.

[6] W. Rampen, "The development of digital displacement technology," *In Proceedings of Bath/ASME FPMC Symposium*, 2010.

[7] P. Johansen, *Tribodynamic Modeling of Digital Fluid Power Motors*. PhD thesis, Department of Energy Technology, Aalborg University, 2014.

[8] D. B. Roemer, *Design and Optimization of Fast Switching Valves for Large Scale Digital Hydraulic Motors*. PhD thesis, 2014. Department of Energy Technology, Aalborg University.

[9] G. Wilfong, M. Batdorff, and J. Lumkes, "Design and dynamic analysis of high speed on/off poppet valves for digital pump/motors," *In Proceedings of the 6th FPNI-PhD Symposium*, 2010.

[10] G. Wilfong, M. Holland, and J. Lumkes, "Design and analysis of pilot operated high speed on/off valves for digital pump/motors," *In Proceedings of the 52nd National Conference on Fluid Power*, 2011.

[11] K. Merrill, M. Holland, and J. Lumkes, "Analysis of digital pump/motor operating strategies," *Proceedings of the 52nd National Conference on Fluid Power*, 2011.

[12] T. Chen and B. Francis, *Optimal Sampled-Data Control Systems*. Springer, 1995.

[13] D. Nesic and L. Grune, "Lyapunov-based continuous-time nonlinear controller redesign for sampled-data implementation," *Automatica, Vol. 14, Issur 7*, 2005.

[14] D. Nesic and A. Teel, "A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models," *IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 49, NO. 7,*, 2004.

[15] J. H. Sandee, W. P. M. H. Heemels, and S. B. F. Hulsenboom, "Analysis and experimental validation of a sensor-based event-driven controller,"

[16] J. H. Sandee, W. P. M. H. Heemels, and S. B. F. Hulsenboom, "Analysis and experimental validation of a sensor-based event-driven controller," *Proceedings of the 2007 American Control Conference*, 2007.

[17] W. P. M. H. Hellems, R. J. A. Gorter, A. van Zijl, P. B. J. vand den Bosch, S. Weiland, W. H. A. Hendrix, and M. R. vonder, "Asynchronous measurement and control: a case study on motor synchronization," *Control Engineering Practice 7, 1467-1482*, 1999.

[18] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control (Volume: 52, Issue: 9)*, 2007.

[19] A. Seuret, "Stability of non-linear systems by means of event-triggered sampling algorithms,"

[20] M. Heikkila and M. Linjama, "Displacement control of a mobile crane using digital hydraulic power management system," *Mechatronics 23(4), pp. 452 - 461*, 2013.

[21] X. Song, "Modeling an active vehicle suspension system with application of digital displacement pump motor," *Proceedings of the ASME 2008 International Design Engineering Techical Conference/Computers and Information in Enegineering Conference, Brooklyn - New York, Vol. 5, pp. 749-753*, 2008.

[22] B. S. R. Armstrong and Q. Yuan, "Multi-level control of hydraulic gerotor motor and pumps," *Proceedings of the 2006 American Control Conference, Minneapolis, Minnesota, USA*, June 2006. Proceedings of the 2006 American Control Conference Minneapolis, Minnesota.

[23] K. U. Sniegucki M, Gottfried M, "Optimal control of digital hydraulic drives using mixed-integer quadratic programming," September 2013. 9th IFAC Symposium on Nonlinear Control System, Toulouse.

[24] P. Johansen, D. B. Roemer, H. C. Pedersen, and T. O. Andersen, "Delta-sigma modulated displacement of a digital fluid power pump," *In Proceedings of the 7th Workshop on Digital Fluid Power*, February 2015 2015. The Seventh Workshop on Digital Fluid Power, Linz, Austria.

[25] P. Johansen, D. B. Roemer, H. C. Pedersen, and T. O. Andersen, "Discrete linear time invariant analysis of digital fluid power pump flow control," *Journal of dynamic system measurement and control, ASME, accepted for publication*, 2016.

[26] P. J. Niels H. Pedersen and T. O. Andersen, "Lqr feedback control development for wind turbines featuring a digital fluid power transmission system," 2016.

[27] P. J. Niels H. Pedersen and T. O. Andersen, "Optimal control development of a wind turbine featuring a digital fluid power transmission," 2016.

[28] N. H. Pedersen, P. Johansen, and T. O. Andersen, "Event-driven control of a speed varying digital displacement machine," *Proceedings of ASME/BATH FPMC Symposium on Fluid Power and Motion Control, Sarasota, Florida, USA*, 2017.

[29] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE control systems magazine*, 2009.

[30] A. R. Teel, R. G. Sanfelice, and R. Goebel, "Hybrid control systems," 2009.

[31] R. G. Sanfelice and A. R. Teel, "Lyapunov analysis of sample-and-hold hybrid feedbacks," 2005+.

[32] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE control systems magazine*, 2009.

[33] J. D. Reiss, "Understanding sigma-delta modulation: The solved and unsolved issues," *Journal Audio Engineering Society, Vol. 56, No. 1/2*, 2008.